Analytical Evaluation of the Hadron Number in Extensive Air Showers for Rising Inelasticity

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A purely analytical method is presented in this paper for evaluating the average number of hadrons (in an air shower) with a given energy as the inelasticity of the cross-section rises. Our result is found to be in excellent agreement with numerical solutions and also with experimental data, particularly in the high energy range.

I. Introduction

In view of its importance for multiparticle production processes in the ultra high energy region the analysis of the hadronic components of extensive air showers (EAS) has become an active field of interest. A crucial point of such an analysis consists in determining the average number of hadrons with a given energy. Unfortunately, most of the studies employ numerical techniques involving a cluttering mass of experimental data and a lot of equipment.

The present paper employs a purely analytical method to determine the average number of hadrons with a given initial energy as the inelasticity of the cross-section rises. In a recent paper [1] a solution for nucleons has been found; in the pion case the diffusion equation itself is somewhat harder but our previous approach works, mutatis mutandis, quite satisfactorily. As in the previous paper [1] we shall follow the rigorous saddle-point method which we developed in a recent work [2] for the inversion of the Mellin transform.

The raison d'être of the present paper will be apparent when one considers the following facts:

In the first place the coupled integro-differential equations that occur in the present problem have prompted one numerical solution after another [3]−[5] although they are by no means intractable analytically. Perusing some of them which are found to be protracted over sixteen odd years, we have undertaken here to vindicate the feasibility of a purely analytical solution. For the general reader in particular an analytical solution is always more suggestive of novel ideas than any cut and dried numerical sheet. Secondly, the prevalent numerical solutions employ "stronger" assumptions than we need even in our wholly analytical approach. In a standard numerical calculation [3] the nucleon interaction length is approximated by a constant (see the paragraph preceding Eq. (2.4) of [3]). In another [4] λ_π and λ_N are accorded identical values throughout. We, however, consider a more general situation in the sense that we do not require λ_π and λ_N to be equal. Of course, the present work too needs some assumption. We assume $a_\pi = a_N$ (see Eqs. (4)−(6) below); but then, this is evidently a more tenable hypothesis than $\lambda_\pi = \lambda_N$.

II. Analytical Evaluation of the Hadron Number

Consider the relevant integro-differential equations (of diffusion):

$$\frac{\partial N}{\partial p} = -\frac{N'}{\lambda_N} + \int_E^{E_0} \frac{N(E', P)}{\lambda_N} W_{NN}(E', E)\, dE', \qquad (1)$$

$$\frac{\partial \Pi}{\partial p} = -\frac{1}{\lambda_\pi}\Pi + \frac{B}{E_p}\Pi + \int_E^{E_0} \frac{N(E', p)}{\lambda_\pi} W_{\pi N}(E', E)\, dE'$$
$$+ \int_E^{E_0} \frac{\Pi(E', p)}{\lambda_\pi} W_{\pi\pi}(E', E)\, dE', \qquad (2)$$

where $\lambda_{in,\pi}$ etc. are written λ_π etc. for simplicity, the rest of the symols being as in [5] and [6].

Assuming $B/E_p \ll 1$ and setting $\Pi = F_\pi e^{-p/\lambda_\pi}$ and $p' = p/\lambda_\pi(E)$, we reduce (2) into:

$$\frac{\partial F_\pi}{\partial p'} = \int_E^{E_0} \frac{\lambda_\pi(E)}{\lambda_N(E')} \exp\{p/\lambda_\pi(E)\} F_N$$

$$\cdot \exp\{-p/\lambda_N(E')\} W_{\pi N}(E', E) \, dE'$$

$$+ \int_E^{E_0} \frac{\lambda_\pi(E)}{\lambda_\pi(E')} e^p \left(\frac{1}{\lambda_\pi(E)} - \frac{1}{\lambda_\pi(E')}\right)$$

$$\cdot F_\pi W_{\pi\pi}(E', E) \, dE', \qquad (3)$$

where we have written

$$N(E', p) = F_N(E', p) \exp\{-p/\lambda_N(E')\}.$$

We next express the rise of inelasticity in the standard form:

$$\sigma_{in} = \sigma^0 \left(1 + a \ln \frac{E}{100}\right),$$

i.e.

$$\frac{1}{\lambda_{\pi,N}} = \left(1 + a_{\pi,N} \ln \frac{E}{100}\right) \frac{1}{\lambda_{\pi,N}^0}. \qquad (4)$$

As noted before, we do not require $\lambda_\pi = \lambda_N$ but use the somewhat more generally valid assumption $\alpha_\pi = \alpha_N = a$ (say).

From (4), writing $X = E/E'$, we obtain

$$\frac{\lambda_\pi(E)}{\lambda_\pi(E')} \cong 1 - a \ln X, \qquad (5)$$

and

$$\frac{\lambda_\pi(E)}{\lambda_N(E')} \cong \frac{\lambda_\pi^0}{\lambda_N^0}(1 - a \ln X). \qquad (6)$$

Also

$$\frac{1}{\lambda_\pi(E)} - \frac{1}{\lambda_N(E')} \cong \frac{1}{\lambda_\pi^0} - \frac{1}{\lambda_N^0} + \frac{1}{\lambda_\pi^0} a \ln X, \qquad (7)$$

the last equation following from a neglect of second order of smallness, viz. $a\left(\frac{1}{\lambda_\pi^0} - \frac{1}{\lambda_N^0}\right)$.

Accordingly (3) reduces to ($w = E/E_0$)

$$\frac{\partial F_\pi}{\partial p'}(E, p') = \frac{\lambda_\pi^0}{\lambda_N^0} \int_w^1 f_{\pi N}(X) \frac{dx}{X^2} (1 - a \ln X) F_N$$

$$\cdot \exp\{a p' \ln X\} \cdot \exp\left\{p'\left(1 - \frac{\lambda_\pi^0}{\lambda_N^0}\right)\right\}$$

$$+ \int_w^1 f_{\pi\pi}(X) \frac{dx}{X^2}(1 - a \ln X) F_\pi$$

$$\cdot \exp\{a p' \ln X\}, \qquad (8)$$

where we have introduced a naive form of Feynman scaling typified by

$$f_{\pi N}(X) = E W_{\pi N}(E, E') = \frac{1}{\sigma_{in}} \int E \frac{\alpha^3 \sigma}{\alpha p^3} \, dp_t^2.$$

Subjecting (8) to Mellin transform (defined by $M_\pi = \int_0^\infty F_\pi E^s \, dE$; $M_N = \int_0^\infty F_N E^s \, dE$), we have for large E_0,

$$\frac{\partial M_\pi}{\partial p'} = \frac{\lambda_\pi^0}{\lambda_N^0} g(s) e^{\varepsilon p'} M_N(s, p') + M_\pi(s, p') g(s), \qquad (9)$$

where

$$g(s) = \int_0^1 x^{s+ap'-1} f(X)(1 - a \ln X) \, dx$$

and

$$\varepsilon = 1 - \frac{\lambda_\pi^0}{\lambda_N^0}.$$

Equation (9) may be considered a linear differential equation for M_π whose solution (ignoring the p' dependence of $g(s)$ which is inconsequential in view of the smallness of a) is given by

$$M_\pi = C e^{p' g(s)} + e^{p' g(s)} \frac{\lambda_\pi^0}{\lambda_N^0}$$

$$\cdot \int g(s) e^{(\varepsilon - g(s)) p'} M_N(s, p') \, dp', \qquad (10)$$

where C is an arbitrary integration constant.

But

$$M_N(s, p') = \int_0^\infty F_N E^s \, dE \cong M_N(s, 0) e^{p' \bar{g}(s)},$$

where $\bar{g}(s)$ is related to the process $NN \to p + X$. Since $M_\pi(s, 0) = 0$, C in (10) can be known.

Also, for $\bar{g}(s) + \varepsilon \ll g(s)$, we have

$$M_\pi \cong \frac{\lambda_\pi^0}{\lambda_N^0} M_N(s, 0) [e^{p' g(s)} - e^{p' \bar{g}(s)}]. \qquad (11)$$

Again,

$$M_N \cong M_N(s, 0) e^{p' \bar{g}(s)}. \qquad (12)$$

Hence

$$M_\pi + \frac{\lambda_\pi^0}{\lambda_N^0} M_N = \frac{\lambda_\pi^0}{\lambda_N^0} M_N(s, 0) e^{p' \bar{g}(s)}. \qquad (13)$$

What is still left is just the inversion of this Mellin-transformed relation. This is found from our previous work of inversion [2] by a comparison. We

accordingly find

$$\Pi + \frac{\lambda_\pi^0}{\lambda_N^0} N = \frac{\lambda_\pi^0}{\lambda_N^0} \frac{C_0}{E} \sqrt{\frac{\pi}{p'}}$$
$$\cdot \exp\{p'\,[g\,(s_0) - 1]\,\bar{s}_0 \ln w\}\,. \qquad (14)$$

Also, from another previous work of ours [1],

$$N = \frac{\bar{C}_0}{E} \sqrt{\frac{\pi}{p''}} \exp\{p\,[\bar{g}\,(s_0) - 1] - \bar{s}_0 \ln W\}\,, \qquad (15)$$

where $p'' = \dfrac{p}{\lambda_N(E)}$ and s_0 is the saddle-point ob-
tained by solving (1). Note that the elegance of the
solution (14) is that the only free parameter here is
C_0; for details of the inversion procedure we allude
to our previous works [1, 2]. From (14) and (15),
then, the total hadron spectrum $\Pi + N$ can easily be
obtained. If, however, we use $\lambda_N^0 \approx \lambda_\pi^0$, (13) alone
would yield the total hadron spectrum:

$$\Pi + N = \frac{C_0}{E} \sqrt{\frac{\pi}{p'}} \exp\{p'\,[g\,(s_0) - 1] - \bar{s}_0 \ln W\}\,, \qquad (16)$$

where s_0 is given by $\left.\dfrac{dg\,(s)}{\alpha s}\right|_{s=s_0} = \dfrac{\ln W}{p'}$ and
$\bar{s}_0 = s_0 - ap' + 1$.

Note that in particular, for the process $NN \rightarrow \Pi^\pm + X$, $f(X)$ can be simulated by the formula [7]
$f(X) \sim A_0\,(a - X)^4$.

Then

$$g\,(s) = \int_0^1 X\,e^{s+ap'-1} f(X)\,(1 - a \ln X)\,\mathrm{d}X$$

$$\cong A_5\,B\,(s + ap',\,s)\,, \qquad (17)$$

where B is the usual beta function. In that event S_0^*
admits of an approximate representation in terms of

beta function as follows:

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} B\,(s + ap',\,5)\right|_{s=s_0} = \frac{\ln W}{A_0\,p}\,. \qquad (18)$$

The last expression can be solved by successive
approximation.

For comparing the result of our calculation with a
standard work [8] we use $a = 0.03$ and obtain excel-
lent agreement as evinced by Fig. 1, which more-
over compares our result with the numerical calcu-
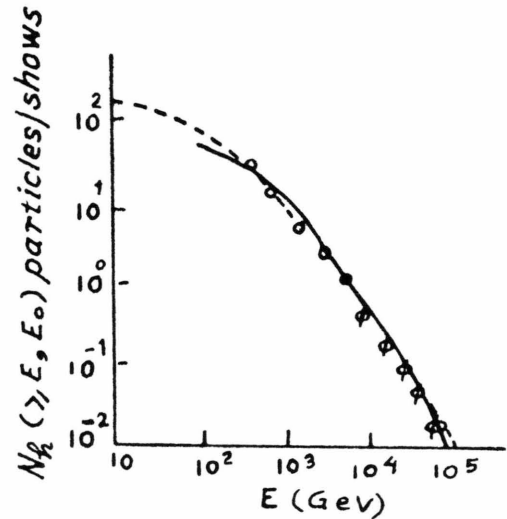lations [5] especially in the higher reaches of the
energy range.

Fig. 1. Energy spectra of hadrons in EAS for $E_0 = 10^6$ GeV
and $a = 0.03$ at $p = 720$ gm/cm^2. ---- Numerical results
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